Combinatorial aspects of random planar triangulations of the disk coupled with an Ising model

Joonas Turunen
ENS de Lyon

joonas.turunen@ens-lyon.fr

Réunion ANR Combiné
8 February 2022

Based on joint research articles with Linxiao Chen (ETH Zurich)
Background

- Ising model is a canonical model of ferromagnetism in statistical mechanics
- Usually defined on a fixed graph
Background

- Ising model is a canonical model of ferromagnetism in statistical mechanics
- Usually defined on a fixed graph
- Introduced by Lenz (1920), solved by Ising in 1d (1924-1925)
- Generalized to higher dimensions by various authors, while most of the interesting rigorous results proven in 2d
Ising model is a canonical model of ferromagnetism in statistical mechanics.

Usually defined on a fixed graph.

Introduced by Lenz (1920), solved by Ising in 1d (1924-1925).

Generalized to higher dimensions by various authors, while most of the interesting rigorous results proven in 2d.

Some remarkable properties in 2d: exact solution and phase transition (Onsager, 1944), Conformal Field Theory (Belavin, Polyakov, Zamolodchikov, 1984 →), conformally invariant scaling limits of interfaces (Smirnov et al, 2010 →).
Ising model on a graph

- Let $G$ be a finite graph, $V(G)$ its vertex set and $E(G)$ its edge set.
- A spin configuration $\sigma$ on $G$ is formally defined as $\sigma = (\sigma_v)_{v \in V(G)} \in \{-1, +1\}^{V(G)}$. 

Assign a Boltzmann measure on spin configurations by

$$P_{\beta G}(\sigma) \propto \prod_{\{v, w\} \in E(G)} e^{\beta \sigma_v \sigma_w}$$

where $\beta$ is called the inverse temperature.

Partition function

$$Z_G(\beta) = \sum_{\sigma} \prod_{\{v, w\} \in E(G)} e^{\beta \sigma_v \sigma_w}$$

The Boltzmann distribution can be reformulated as

$$P_\nu G(\sigma) \propto \nu \# \{\{v, w\} \in E(G) : \sigma_v = \sigma_w\}$$

In particular, $\beta > 0 \Leftrightarrow \nu > 1$. In this regime, the model is called ferromagnetic, on which we concentrate in the sequel.
Let $G$ be a finite graph, $V(G)$ its vertex set and $E(G)$ its edge set.

A spin configuration $\sigma$ on $G$ is formally defined as
$\sigma = (\sigma_v)_{v \in V(G)} \in \{-1, +1\}^{V(G)}$.

Assign a Boltzmann measure on spin configurations by

$$\mathbb{P}_G^\beta(\sigma) \propto \prod_{\{v, w\} \in E(G)} e^{\beta \sigma_v \sigma_w}$$

where $\beta$ is called the inverse temperature.
Ising model on a graph

- Let $G$ be a finite graph, $V(G)$ its vertex set and $E(G)$ its edge set.
- A spin configuration $\sigma$ on $G$ is formally defined as $\sigma = (\sigma_v)_{v \in V(G)} \in \{-1, +1\}^{V(G)}$.
- Assign a Boltzmann measure on spin configurations by
  $$P_G^\beta(\sigma) \propto \prod_{\{v, w\} \in E(G)} e^{\beta \sigma_v \sigma_w}$$
  where $\beta$ is called the inverse temperature.
- Partition function $Z_G(\beta) = \sum_\sigma \prod_{\{v, w\} \in E(G)} e^{\beta \sigma_v \sigma_w}$
Ising model on a graph

- Let $G$ be a finite graph, $V(G)$ its vertex set and $E(G)$ its edge set.

- A *spin configuration* $\sigma$ on $G$ is formally defined as $\sigma = (\sigma_v)_{v \in V(G)} \in \{-1, +1\}^{V(G)}$.

- Assign a Boltzmann measure on spin configurations by

$$
P^\beta_G(\sigma) \propto \prod_{\{v, w\} \in E(G)} e^{\beta \sigma_v \sigma_w}
$$

where $\beta$ is called the *inverse temperature*.

- Partition function $Z_G(\beta) = \sum_{\sigma} \prod_{\{v, w\} \in E(G)} e^{\beta \sigma_v \sigma_w}$

- The Boltzmann distribution can be reformulated as

$$
P^\nu_G(\sigma) \propto \nu^{\# \{\{v, w\} \in E(G) : \sigma_v = \sigma_w\}}
$$

- In particular, $\beta > 0 \iff \nu > 1$. In this regime, the model is called *ferromagnetic*, on which we concentrate in the sequel.
An example in 2d with a planar embedding
Ising model on random ("dynamical") lattices

- Dates back to the work of Kazakov (1986) and Boulatov - Kazakov (1987)
- Original physics motivations: "Liouville Quantum Gravity coupled with matter" (Polyakov 1981); quantum vs Euclidean critical exponents via the KPZ-relation (Knizhnik-Polyakov-Zamolodchikov 1988)
Ising model on random ("dynamical") lattices

- Dates back to the work of Kazakov (1986) and Boulatov - Kazakov (1987)
- Original physics motivations: "Liouville Quantum Gravity coupled with matter" (Polyakov 1981); quantum vs Euclidean critical exponents via the KPZ-relation (Knizhnik-Polyakov-Zamolodchikov 1988)
- The above works already revealed a critical behavior different from the pure gravity universality class
Ising model on random ("dynamical") lattices

- Dates back to the work of Kazakov (1986) and Boulatov - Kazakov (1987)
- Original physics motivations: "Liouville Quantum Gravity coupled with matter" (Polyakov 1981); quantum vs Euclidean critical exponents via the KPZ-relation (Knizhnik-Polyakov-Zamolodchikov 1988)
- The above works already revealed a critical behavior different from the pure gravity universality class
- In the language of modern mathematics: random planar maps coupled with an (annealed) Ising model
Ising model on random ("dynamical") lattices

- Dates back to the work of Kazakov (1986) and Boulatov - Kazakov (1987)
- Original physics motivations: "Liouville Quantum Gravity coupled with matter" (Polyakov 1981); quantum vs Euclidean critical exponents via the KPZ-relation (Knizhnik-Polyakov-Zamolodchikov 1988)
- The above works already revealed a critical behavior different from the pure gravity universality class
- In the language of modern mathematics: random planar maps coupled with an (annealed) Ising model
- We want to find a critical behavior of the model which differs from the "universality class of the Brownian map"
Planar maps

- A \textit{planar map} is a connected multigraph properly embedded on $S^2$, modulo orientation preserving homeomorphisms of $S^2$. 
A **planar map** is a connected multigraph properly embedded on \( S^2 \), modulo orientation preserving homeomorphisms of \( S^2 \).

We distinguish an oriented edge (or corner) of the map which we call the **root**; this provides a trivial automorphism group.
A planar map is a connected multigraph properly embedded on $S^2$, modulo orientation preserving homeomorphisms of $S^2$. We distinguish an oriented edge (or corner) of the map which we call the root; this provides a trivial automorphism group. We also distinguish a face to which the root is incident to and call it the external face.
Planar maps

- A planar map is a connected multigraph properly embedded on $S^2$, modulo orientation preserving homeomorphisms of $S^2$.
- We distinguish an oriented edge (or corner) of the map which we call the root; this provides a trivial automorphism group.
- We also distinguish a face to which the root is incident to and call it the external face.
- A rooted planar map is a triangulation of the $m$-gon if all its internal faces are triangles and the external face has degree $m$ and no pinch-points.
A planar map is a connected multigraph properly embedded on $S^2$, modulo orientation preserving homeomorphisms of $S^2$. We distinguish an oriented edge (or corner) of the map which we call the root; this provides a trivial automorphism group. We also distinguish a face to which the root is incident to and call it the external face.

A rooted planar map $t$ is a triangulation of the $m$-gon if all its internal faces are triangles and the external face has degree $m$ and no pinch-points.
Add to each internal face (or vertex) a spin, either + or −.
Add to each internal face (or vertex) a spin, either $+$ or $-$.  

*Dobrushin boundary conditions*: the spins outside the boundary (resp. on the boundary) are fixed by a sequence of the form $+^p -^q$ counterclockwise from the root.
• From now on, we consider the model when the spins are on the faces of the triangulation.

• Let $\mathcal{F}(t)$ be the set of internal faces of $t$. 

root corner color code:
\[= \text{spin}+\]
\[= \text{spin}–\]
From now on, we consider the model when the spins are on the faces of the triangulation.

Let $\mathcal{F}(t)$ be the set of internal faces of $t$.

Denote a spin configuration by $\sigma$. 
From now on, we consider the model when the spins are on the faces of the triangulation.

Let $\mathcal{F}(t)$ be the set of internal faces of $t$.

Denote a spin configuration by $\sigma$.

An edge is called *monochromatic* if it separates two faces with the same spin. Let $\mathcal{E}(t, \sigma)$ be the set of monochromatic edges in $(t, \sigma)$.

\[ (a) \quad (t, \sigma) \in \mathcal{B}T_{3,4}, \quad |\mathcal{F}(t)| = 19 \]

\[ (b) \quad (t, \sigma) \in \mathcal{B}T_{3,4}, \quad |\mathcal{E}(t, \sigma)| = 18 \]

(color code: $\purple = \text{spin } +$, $\turquoise = \text{spin } -$)
Partition functions

Partition function

\[ z_{p,q}(t, \nu) = \sum_{(t, \sigma) \in \mathcal{B}T_{p,q}} \nu |\mathcal{E}(t, \sigma)| t |\mathcal{F}(t)| , \]

where

- \( \mathcal{B}T_{p,q} \) is the set of triangulations of the \((p + q)\)-gon together with an Ising-configuration on interior faces and a Dobrushin boundary condition \(+p−q\).
Partition functions

Partition function

\[ z_{p,q}(t, \nu) = \sum_{(t,\sigma) \in \mathcal{BT}_{p,q}} \nu |E(t,\sigma)| t |F(t)|, \]

where

- \( \mathcal{BT}_{p,q} \) is the set of triangulations of the \((p + q)\)-gon together with an Ising-configuration on interior faces and a Dobrushin boundary condition \(+^p -^q\).

Generating function

\[ Z(u, v; t, \nu) = \sum_{p,q \geq 0} z_{p,q}(t, \nu) u^p v^q \]
Theorem [Chen, T., 2020]

For every $\nu > 1$, the GF $Z(u, v; t, \nu)$ is an algebraic function having a rational parametrization

$$
t^2 = \hat{T}(S, \nu), \quad t \cdot u = \hat{U}(H; S, \nu), \quad t \cdot v = \hat{U}(K; S, \nu)
$$

$$
Z(u, v; t, \nu) = \hat{Z}(H, K; S, \nu),
$$

where $\hat{T}$, $\hat{U}$ and $\hat{Z}$ are rational functions with explicit expressions.
Theorem [Chen, T., 2020]

For every $\nu > 1$, the GF $Z(u, v; t, \nu)$ is an algebraic function having a rational parametrization

$$
t^2 = \hat{T}(S, \nu), \quad t \cdot u = \hat{U}(H; S, \nu), \quad t \cdot v = \hat{U}(K; S, \nu)
$$

$$
Z(u, v; t, \nu) = \hat{Z}(H, K; S, \nu),
$$

where $\hat{T}$, $\hat{U}$ and $\hat{Z}$ are rational functions with explicit expressions.

This theorem indicates that the model is "exactly solvable": various observables (e.g. the free energy) can be explicitly computed at least in some scaling limits from the expression of the generating function!
Proof ingredients: peeling and functional equation for \( Z \)

\[
\begin{aligned}
q_{p+1} + & \nu t \begin{pmatrix}
qu & p+2 \\
p & \end{pmatrix} +
\begin{pmatrix}
qu' & p+1 \\
1 & 1 \\
k & \end{pmatrix} +
\begin{pmatrix}
qu-k & \end{pmatrix} -
\begin{pmatrix}
q & 1 \\
1 & p \\
L & \end{pmatrix} \\
\end{aligned}
\]

\[
+ t \begin{pmatrix}
q+2 & p \\
p & \end{pmatrix} +
\begin{pmatrix}
qu-1 & p+1 \\
1 & 1 \\
R & \end{pmatrix} +
\begin{pmatrix}
qu-k & \end{pmatrix} -
\begin{pmatrix}
q & 1 \\
1 & p \\
L & \end{pmatrix} \\
\end{aligned}
\]

\[
+ \nu \cdot \delta_{p,1} \delta_{q,0} + \delta_{p,0} \delta_{q,1}
\]
Proof ingredients: peeling and functional equation for $Z$

$$\nu t \left( C^+ u_{p+2} + C^- u'_{1,q} + R^+_k u'_{q+k} + L^-_k u_{q+k} - L^+_q = R^+_p \right)$$

$$\nu \cdot \delta_{p,1} \delta_{q,0} + \delta_{p,0} \delta_{q,1}$$

$$z_{p+1,q} = \nu t \left( z_{p+2,q} + \sum_{p_1+p_2=p} z_{p_1+1,0} z_{p_2+1,q} + \sum_{q_1+q_2=q} z_{1,q_1} z_{p+1,q_2} - z_{p+1,0} z_{1,q} \right)$$

$$+ t \left( z_{p,q+2} + \sum_{p_1+p_2=p} z_{1,p_1} z_{p_2,q+1} + \sum_{q_1+q_2=q} z_{0,q_1+1} z_{p,q_2+1} - z_{p,1} z_{0,q+1} \right)$$

$$+ \nu \delta_{p,1} \delta_{q,0} + \delta_{p,0} \delta_{q,1}$$  (1)
Summing over $p, q$, we obtain a linear equation for $Z(u, v)$, and interchanging the roles of $p$ and $q$ gives a linear system

$$
\begin{bmatrix}
\Delta_u Z(u, v) \\
\Delta_v Z(v, u)
\end{bmatrix}
$$

(2)

$$
= \begin{bmatrix}
u & 1 \
1 & \nu
\end{bmatrix} \begin{bmatrix}
u + t (\Delta^2_u Z(u) + (\Delta Z_0(u) + Z_1(v)) \Delta_u Z(u) - \Delta Z_0(u)Z_1(v)) \\
\nu + t (\Delta^2_v Z(v) + (\Delta Z_0(v) + Z_1(u)) \Delta_v Z(v) - \Delta Z_0(v)Z_1(u))
\end{bmatrix},
$$

where

$$
Z_k(u) := [\nu^k] Z(u, v), \quad \Delta_u Z(u, v) = \frac{Z(u, v) - Z_0(v)}{u},
$$

$$
\Delta Z_0(u) = \frac{Z_0(u) - 1}{u}, \quad \Delta^2_u Z(u, v) = \frac{Z(u, v) - Z_0(v) - uZ_1(v)}{u^2}
$$

and so on.
It turns out that $Z_1$ can be eliminated, and thus we obtain a rational expression

$$Z(u, v) = \frac{R_1(u, v, Z_0(u), Z_0(v))}{R_2(u, v, Z_0(u), Z_0(v))}.$$  \hspace{1cm} (3)

where $R_1$, $R_2$ are explicit polynomials.

Besides, we obtain a functional equation

$$\mathcal{P}(Z_0(u), u, z_1, z_3; t, \nu) = 0,$$ \hspace{1cm} (4)

where $\mathcal{P}$ is an explicit polynomial and $z_k := z_{k,0}$. 

Luckily, we can obtain rational parametrizations for $t, z_1$ and $z_3$ by simple duality with the model in [Bernardi, Bousquet-Melou [1]]. This can also be done directly from (4) (more messy).

Applying (computer) algebra we find an explicit RP for $u$ and $Z_0(u)$ for any given $\nu > 1$. 
It turns out that $Z_1$ can be eliminated, and thus we obtain a rational expression

$$Z(u, v) = \frac{R_1(u, v, Z_0(u), Z_0(v))}{R_2(u, v, Z_0(u), Z_0(v))}.$$  \hspace{1cm} (3)

where $R_1$, $R_2$ are explicit polynomials.

- Besides, we obtain a functional equation

$$\mathcal{P}(Z_0(u), u, z_1, z_3; t, \nu) = 0,$$  \hspace{1cm} (4)

where $\mathcal{P}$ is an explicit polynomial and $z_k := z_{k,0}$.

- Luckily, we can obtain rational parametrizations for $t$, $z_1$ and $z_3$ by simple duality with the model in [Bernardi, Bousquet-Melou [1]]. This can also be done directly from (4) (more messy).

- Applying (computer) algebra we find an explicit RP for $u$ and $Z_0(u)$ for any given $\nu > 1$. 
Proposition (Bernardi, Bousquet-Mélou [1])

There is a continuous decreasing function $\tau : (0, \infty) \to (0, \infty)$ for which

$$[t^n] z_{1,0}(t, \nu) \sim_{n \to \infty} \begin{cases} c(\nu) \tau(\nu)^{-n} n^{-5/2} & \text{if } \nu \neq \nu_c \\ c(\nu_c) t_c^{-n} n^{-7/3} & \text{if } \nu = \nu_c \end{cases}$$

where $\nu_c = 1 + 2\sqrt{7}$ and $t_c = \tau(\nu_c) = \frac{\sqrt{5\sqrt{35-11\sqrt{7}}}}{28.6^{3/2}} = 0.0131\ldots$
Proposition (Bernardi, Bousquet-Mélou [1])

There is a continuous decreasing function \( \tau : (0, \infty) \to (0, \infty) \) for which

\[
[t^n]z_{1,0}(t, \nu) \sim_{n \to \infty} \begin{cases} 
    c(\nu)\tau(\nu)^{-n}n^{-5/2} & \text{if } \nu \neq \nu_c \\
    c(\nu_c)t_c^{-n}n^{-7/3} & \text{if } \nu = \nu_c 
\end{cases}
\]

where \( \nu_c = 1 + 2\sqrt{7} \) and \( t_c = \tau(\nu_c) = \frac{\sqrt{5}\sqrt{35-11\sqrt{7}}}{28.63^{3/2}} = 0.0131 \ldots \).

- Relying on the above result, we identify a critical line \((\nu, \tau(\nu))\) for \( \nu > 1 \), and a unique critical point \((\nu_c, t_c)\) on the critical line at which a phase transition occurs.
- \( t_c(\nu) := \tau(\nu) \) is simply the radius of convergence of \( z_{1,0}(t, \nu) \) for a fixed \( \nu > 1 \).
Theorem [Chen, T., 2020]

For $\nu > 1$, 

$$z_{p,q}(t_c(\nu), \nu) \sim \frac{a_p(\nu)}{\Gamma(-\alpha_0)} u_c(\nu)^{-q} q^{-(\alpha_0+1)} \quad \text{as } q \to \infty;$$

$$a_p(\nu) \sim \frac{b(\nu)}{\Gamma(-\alpha_1)} u_c(\nu)^{-p} p^{-(\alpha_1+1)} \quad \text{as } p \to \infty;$$

$$z_{p,q}(t_c(\nu), \nu) \sim \frac{b(\nu) \cdot c(q/p)}{\Gamma(-\alpha_0)\Gamma(-\alpha_1)} u_c(\nu)^{-(p+q)} p^{-(\alpha_2+2)} \quad \text{as } p, q \to \infty$$

while $q/p \in [\lambda_{\min}, \lambda_{\max}]$ where $0 < \lambda_{\min} < \lambda_{\max} < \infty$.

The perimeter exponents are determined by the following table:

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$(1, \nu_c)$</th>
<th>{\nu_c}</th>
<th>$(\nu_c, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_0$</td>
<td>$3/2$</td>
<td>$4/3$</td>
<td>$3/2$</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>$-1$</td>
<td>$1/3$</td>
<td>$3/2$</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>$1/2$</td>
<td>$5/3$</td>
<td>$3$</td>
</tr>
</tbody>
</table>
Proof ideas

We want to understand the singularity structure of $Z(u, v; \nu)$, which boils down to understanding the one of the RP $(\hat{Z}(H, K; R), \hat{U}(H; R), \hat{U}(K; R))$ with $\nu = \hat{\nu}(R)$. This involves:

- Identifying the singularity $u_c(\nu)$ by looking at the critical points of $\hat{U}$, and showing that $\hat{U}$ defines a conformal bijection between the domains of convergences of $\hat{Z}$ and $Z$ around the origin.
- Showing that $u_c(\nu)$ is the unique dominant singularity of $Z$, which in particular involves showing that $\hat{Z}$ has only one pole which is mapped to $\partial D(0, u_c(\nu))^2$ under the aforementioned conformal bijection.
- Deducing that $Z$ is holomorphic in a product of ∆-domains, which roughly means that it is amenable to transfer theorems of analytic combinatorics (see the book of Flajolet and Sedgewick).
Proof ideas

We want to understand the singularity structure of \( Z(u, v; \nu) \), which boils down to understanding the one of the RP \( (\hat{Z}(H, K; R), \hat{U}(H; R), \hat{U}(K; R)) \) with \( \nu = \hat{\nu}(R) \). This involves:

- Identifying the singularity \( u_c(\nu) \) by looking at the critical points of \( \hat{U} \), and showing that \( \hat{U} \) defines a conformal bijection between the domains of convergences of \( \hat{Z} \) and \( Z \) around the origin.
Proof ideas

We want to understand the singularity structure of $Z(u, \nu; \nu)$, which boils down to understanding the one of the RP $(\hat{Z}(H, K; R), \hat{U}(H; R), \hat{U}(K; R))$ with $\nu = \hat{\nu}(R)$. This involves:

- Identifying the singularity $u_c(\nu)$ by looking at the critical points of $\hat{U}$, and showing that $\hat{U}$ defines a conformal bijection between the domains of convergences of $\hat{Z}$ and $Z$ around the origin.

- Showing that $u_c(\nu)$ is the unique dominant singularity of $Z$, which in particular involves showing that $\hat{Z}$ has only one pole which is mapped to $\partial D(0, u_c(\nu))^2$ under the aforementioned conformal bijection.
We want to understand the singularity structure of $Z(u, v; \nu)$, which boils down to understanding the one of the RP $(\hat{Z}(H, K; R), \hat{U}(H; R), \hat{U}(K; R))$ with $\nu = \hat{\nu}(R)$. This involves:

- Identifying the singularity $u_c(\nu)$ by looking at the critical points of $\hat{U}$, and showing that $\hat{U}$ defines a conformal bijection between the domains of convergences of $\hat{Z}$ and $Z$ around the origin.

- Showing that $u_c(\nu)$ is the unique dominant singularity of $Z$, which in particular involves showing that $\hat{Z}$ has only one pole which is mapped to $\partial D(0, u_c(\nu))^2$ under the aforementioned conformal bijection.

- Deducing that $Z$ is holomorphic in a product of $\Delta$-domains, which roughly means that it is amenable to transfer theorems of analytic combinatorics (see the book of Flajolet and Sedgewick).
The previous results allow us to write local expansions of $Z(u, \nu; \nu)$ around $u, \nu = u_c(\nu)$, for which we apply the Cauchy integral formula to find the asymptotics of $z_{p,q}(t_c(\nu), \nu)$.

The local expansions depend on the temperature regime, hence the different critical exponents and a (combinatorial) phase transition.
A "geometric" reminder

\[(a) \quad (t, \sigma) \in \mathcal{B}T_{3,4}
\]

\[|\mathcal{F}(t)| = 19\]

\[|\mathcal{E}(t, \sigma)| = 18\]

root corner

\[\rho\]

(a)

(b)

color code:

= spin +

= spin -
Boltzmann distribution

Definition

The Boltzmann Ising-triangulation of the \((p, q)\)-gon is a random variable having the law

\[
\mathbb{P}_{p, q}(t, \sigma) = \frac{t_c(\nu)^{\text{Vol}(t) \nu t_c(\nu)}}{z_{p, q}(t_c(\nu), \nu)},
\]

\((t, \sigma) \in BT_{p, q}\).
Definition

The Boltzmann Ising-triangulation of the \((p, q)\)-gon is a random variable having the law

\[
P^\nu_{p,q}(t, \sigma) = \frac{t_c(\nu)^{\text{Vol}(t)\nu} \mathcal{E}(t, \sigma)}{z_{p,q}(t_c(\nu), \nu)},
\]

\((t, \sigma) \in \mathcal{B}T_{p,q}\).

In the previous example,

\(|\mathcal{F}(t)| = 19,\]
\(|\mathcal{E}(t, \sigma)| = 18\) and

\[
P^\nu_{p,q}(t, \sigma) = \frac{t_c(\nu)^{19} \nu^{18}}{z_{3,4}(t_c(\nu), \nu)}.
\]
A glimpse of random geometry; a phase transition

Figure: The local limits \((p, q \to \infty)\) in the high temperature and the low temperature regimes.

Figure: The two local limits at the critical temperature.
Interfaces at the critical temperature

**Figure:** Spin cluster interfaces when the spins are on faces.

**Figure:** The unique infinite interface when the spins are on vertices.
A closely related work

- Albenque, Ménard and Schaeffer [2] considered the set of triangulations of the sphere of size $n$ decorated with an Ising model on the vertices.

- After generalizing [1], they show the local convergence of such triangulations when $n \to \infty$.

- In a recent preprint [3], Albenque and Ménard apply rational parametrizations (including a part of our method) to study the critical perimeter and volume exponents of the spin cluster of the origin.
Near-critical regime ($|\nu - \nu_c| \propto p^{-\beta}$)

Universality (more general lattices)

More general boundary conditions, yielding to recursion on the generating functions

Many probabilistic aspects (scaling limits, relations to Liouville Quantum Gravity and Schramm-Loewner Evolutions,...)

Applications and generalizations of the methods to other statistical mechanics models
This talk was based on:


Related works:


Merci beaucoup!