Combinatorial aspects of random planar triangulations of the disk coupled with an Ising model

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Based on joint research articles with Linxiao Chen (ETH Zurich) [https://doi.org/10.1007/s00220-019-03672-5], [arXiv:2003.09343]

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- Generalized to higher dimensions by various authors, while most of the interesting rigorous results proven in 2d
- Some remarkable properties in 2d: exact solution and phase transition (Onsager, 1944), Conformal Field Theory (Belavin, Polyakov, Zamolodchikov, 1984 $\longrightarrow$), conformally invariant scaling limits of interfaces (Smirnov et al, 2010 $\longrightarrow$)

Ising model on a graph

- Let G be a finite graph, $V(G)$ its vertex set and $E(G)$ its edge set.
- A *spin configuration* σ on G is formally defined as $\sigma = (\sigma_v)_{v \in V(G)} \in \{-1, +1\}^{V(G)}$.

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$$\mathbb{P}_G^{\beta}(\sigma) \propto \prod_{\{v,w\} \in E(G)} e^{\beta \sigma_v \sigma_w}$$

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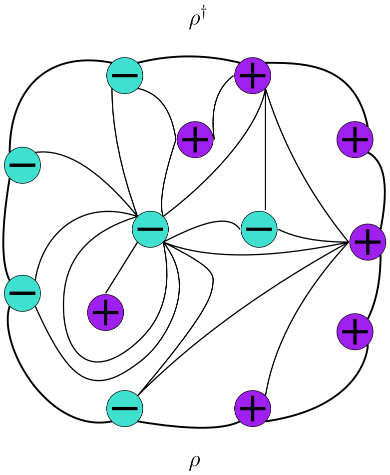
- Partition function $Z_G(\beta) = \sum_\sigma \prod_{\{v,w\} \in E(G)} e^{\beta \sigma_v \sigma_w}$
- The Boltzmann distribution can be reformulated as $\mathbb{P}_G^{\nu}(\sigma) \propto \nu^{\#\{\{v,w\} \in E(G) \,:\, \sigma_v = \sigma_w\}}$
- In particular, $\beta > 0 \Leftrightarrow \nu > 1$. In this regime, the model is called *ferromagnetic*, on which we concentrate in the sequel.

An example in 2d with a planar embedding



Ising model on random ("dynamical") lattices

- Dates back to the work of Kazakov (1986) and Boulatov - Kazakov (1987)
- Original physics motivations: "Liouville Quantum Gravity coupled with matter" (Polyakov 1981); quantum vs Euclidean *critical exponents* via the KPZ-relation (Knizhnik-Polyakov-Zamolodchikov 1988)

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- In the language of modern mathematics: random planar maps coupled with an (annealed) Ising model
- We want to find a critical behavior of the model which differs from the "universality class of the Brownian map"

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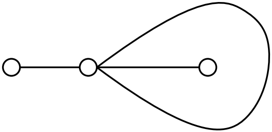

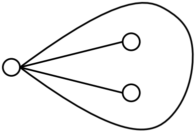

Ising-triangulations

- Add to each internal face (or vertex) a spin, either $+$ or $-$.

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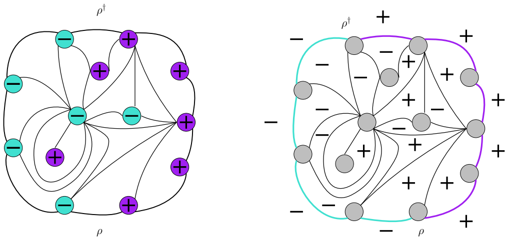
- *Dobrushin boundary conditions*: the spins *outside* the boundary (resp. on the boundary) are fixed by a sequence of the form $+^p-^q$ counterclockwise from the root.



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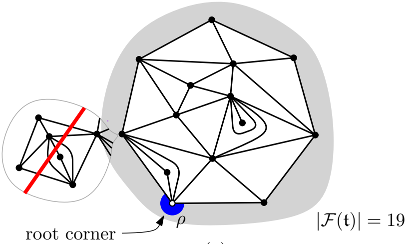
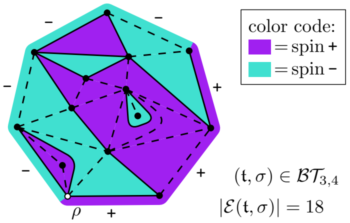
- Denote a spin configuration by σ.
- An edge is called *monochromatic* if it separates two faces with the same spin. Let $\mathcal{E}(\mathfrak{t},\sigma)$ be the set of monochromatic edges in $(\mathfrak{t},\sigma)$.



(a)



(b)

Partition functions

Partition function

$$z_{p,q}(t,\nu) = \sum_{(\mathfrak{t},\sigma)\in\mathcal{BT}_{p,q}} \nu^{|\mathcal{E}(\mathfrak{t},\sigma)|} t^{|\mathcal{F}(\mathfrak{t})|},$$

where

- $\mathcal{BT}_{p,q}$ is the set of triangulations of the $(p+q)$-gon together with an Ising-configuration on interior faces and a Dobrushin boundary condition $+^p -^q$.

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Generating function

$$Z(u,v;t,\nu) = \sum_{p,q\geq 0} z_{p,q}(t,\nu)\, u^p v^q$$

Theorem [Chen, T., 2020]

For every $\nu > 1$, the GF $Z(u, v; t, \nu)$ is an algebraic function having a rational parametrization

$$t^2 = \hat{T}(S, \nu), \qquad t \cdot u = \hat{U}(H; S, \nu), \qquad t \cdot v = \hat{U}(K; S, \nu)$$
$$Z(u, v; t, \nu) = \hat{Z}(H, K; S, \nu)\,,$$

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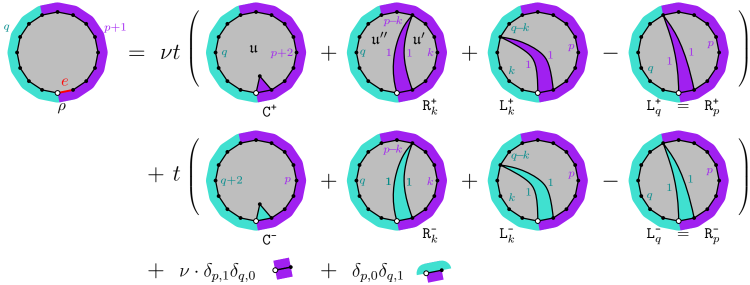
where $\hat{T}$*,* $\hat{U}$ *and* $\hat{Z}$ *are rational functions with explicit expressions.*

This theorem indicates that the model is "exactly solvable": various observables (eg. the free energy) can be explicitly computed at least in some scaling limits from the expression of the generating function!

Proof ingredients: peeling and functional equation for Z



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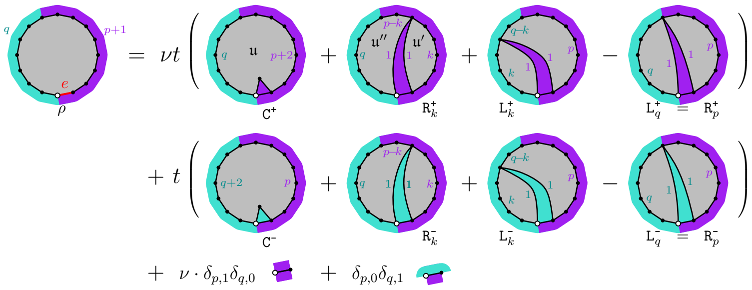

$$z_{p+1,q} = \nu t \left(z_{p+2,q} + \sum_{p_1+p_2=p} z_{p_1+1,0}\, z_{p_2+1,q} + \sum_{q_1+q_2=q} z_{1,q_1}\, z_{p+1,q_2} - z_{p+1,0}\, z_{1,q} \right)$$

$$+\, t \left(z_{p,q+2} + \sum_{p_1+p_2=p} z_{p_1,1}\, z_{p_2,q+1} + \sum_{q_1+q_2=q} z_{0,q_1+1}\, z_{p,q_2+1} - z_{p,1}\, z_{0,q+1} \right)$$

$$+\, \nu\, \delta_{p,1}\, \delta_{q,0} + \delta_{p,0}\, \delta_{q,1} \qquad (1)$$

Summing over p, q, we obtain a linear equation for $Z(u, v)$, and interchanging the roles of p and q gives a linear system

$$\begin{bmatrix} \Delta_u Z(u,v) \\ \Delta_v Z(v,u) \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} \nu & 1 \\ 1 & \nu \end{bmatrix} \begin{bmatrix} u + t\left(\Delta_u^2 Z(u) + (\Delta Z_0(u) + Z_1(v))\,\Delta_u Z(u) - \Delta Z_0(u) Z_1(v)\right) \\ v + t\left(\Delta_v^2 Z(v) + (\Delta Z_0(v) + Z_1(u))\,\Delta_v Z(v) - \Delta Z_0(v) Z_1(u)\right) \end{bmatrix},$$

where

$$Z_k(u) := [v^k] Z(u,v), \quad \Delta_u Z(u,v) = \frac{Z(u,v) - Z_0(v)}{u},$$

$$\Delta Z_0(u) = \frac{Z_0(u) - 1}{u}, \quad \Delta_u^2 Z(u,v) = \frac{Z(u,v) - Z_0(v) - u Z_1(v)}{u^2}$$

and so on.

It turns out that Z_1 can be eliminated, and thus we obtain a rational expression

$$Z(u,v) = \frac{R_1(u,v,Z_0(u),Z_0(v))}{R_2(u,v,Z_0(u),Z_0(v))}. \tag{3}$$

where R_1, R_2 are explicit polynomials.

- Besides, we obtain a functional equation

$$\mathcal{P}(Z_0(u),u,z_1,z_3;t,\nu) = 0, \tag{4}$$

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- Luckily, we can obtain rational parametrizations for t, z_1 and z_3 by simple duality with the model in [Bernardi, Bousquet-Melou [1]]. This can also be done directly from (4) (more messy).

- Applying (computer) algebra we find an explicit RP for u and $Z_0(u)$ for any given $\nu > 1$.

Critical line

Proposition (Bernardi, Bousquet-Mélou [1])

There is a continuous decreasing function $\tau : (0,\infty) \to (0,\infty)$ *for which*

$$[t^n]z_{1,0}(t,\nu) \sim_{n\to\infty} \begin{cases} c(\nu)\tau(\nu)^{-n}n^{-5/2} & \text{if } \nu \neq \nu_c \\ c(\nu_c)t_c^{-n}n^{-7/3} & \text{if } \nu = \nu_c \end{cases}$$

where $\nu_c = 1 + 2\sqrt{7}$ *and* $t_c = \tau(\nu_c) = \frac{\sqrt{5}\sqrt{35-11\sqrt{7}}}{28\cdot 6^{3/2}} = 0.0131\ldots$

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- Relying on the above result, we identify a critical line $(\nu, \tau(\nu))$ for $\nu > 1$, and a unique critical point (ν_c, t_c) on the critical line at which a *phase transition* occurs.
- $t_c(\nu) := \tau(\nu)$ is simply the radius of convergence of $z_{1,0}(t, \nu)$ for a fixed $\nu > 1$.

Theorem [Chen, T., 2020]

For $\nu > 1$,

$$z_{p,q}(t_c(\nu),\nu) \sim \frac{a_p(\nu)}{\Gamma(-\alpha_0)} u_c(\nu)^{-q} \; q^{-(\alpha_0+1)} \qquad \text{as } q \to \infty;$$

$$a_p(\nu) \sim \frac{b(\nu)}{\Gamma(-\alpha_1)} u_c(\nu)^{-p} p^{-(\alpha_1+1)} \qquad \text{as } p \to \infty;$$

$$z_{p,q}(t_c(\nu),\nu) \sim \frac{b(\nu)\cdot c(q/p)}{\Gamma(-\alpha_0)\Gamma(-\alpha_1)} u_c(\nu)^{-(p+q)} p^{-(\alpha_2+2)} \quad \text{as } p,q \to \infty$$

$$\text{while } q/p \in [\lambda_{\min}, \lambda_{\max}] \text{ where } 0 < \lambda_{\min} < \lambda_{\max} < \infty.$$

The perimeter exponents are determined by the following table:

$\nu \in$	$(1,\nu_c)$	$\{\nu_c\}$	(ν_c,∞)
α_0	$3/2$	$4/3$	$3/2$
α_1	-1	$1/3$	$3/2$
α_2	$1/2$	$5/3$	3

Proof ideas

We want to understand the singularity structure of $Z(u, v; \nu)$, which boils down to understanding the one of the RP $(\hat{Z}(H, K; R), \hat{U}(H; R), \hat{U}(K; R))$ with $\nu = \hat{\nu}(R)$. This involves:

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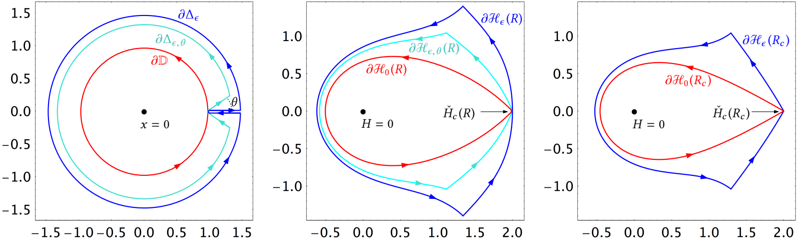
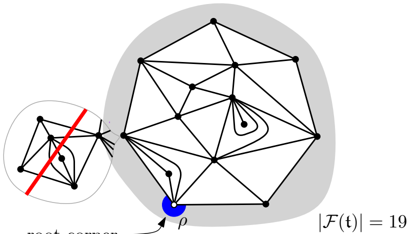
- Showing that $u_c(\nu)$ is the unique dominant singularity of Z, which in particular involves showing that $\hat{Z}$ has only one pole which is mapped to $\partial D(0, u_c(\nu))^2$ under the aforementioned conformal bijection.
- Deducing that Z is holomorphic in a product of Δ-domains, which roughly means that it is amenable to transfer theorems of *analytic combinatorics* (see the book of Flajolet and Sedgewick).

- The previous results allow us to write local expansions of $Z(u, v; \nu)$ around $u, v = u_c(\nu)$, for which we apply the Cauchy integral formula to find the asymptotics of $z_{p,q}(t_c(\nu), \nu)$.
- The local expansions depend on the temperature regime, hence the different critical exponents and a (combinatorial) phase transition.

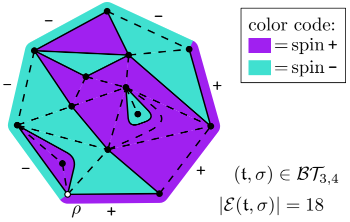


(a) (b) (c)

A "geometric" reminder



(a)



(b)

Boltzmann distribution

Definition

The Boltzmann Ising-triangulation of the (p, q)-gon is a random variable having the law

$$\mathbb{P}_{p,q}^{\nu}(\mathfrak{t}, \sigma) = \frac{t_c(\nu)^{\mathrm{Vol}(\mathfrak{t})} \nu^{\mathcal{E}(\mathfrak{t},\sigma)}}{z_{p,q}(t_c(\nu), \nu)},$$

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In the previous example,
$|\mathcal{F}(\mathfrak{t})| = 19$,
$|\mathcal{E}(\mathfrak{t}, \sigma)| = 18$ and

$$\mathbb{P}^{\nu}_{p,q}(\mathfrak{t}, \sigma) = \frac{t_c(\nu)^{19} \nu^{18}}{z_{3,4}(t_c(\nu), \nu)}$$

A glimpse of random geometry; a phase transition

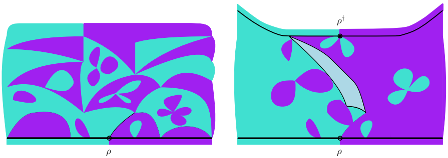

Figure: The local limits ($p, q \to \infty$) in the high temperature and the low temperature regimes.

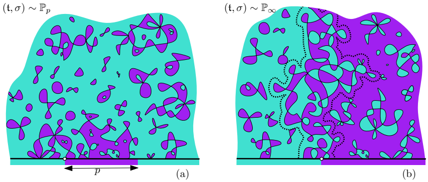

Figure: The two local limits at the critical temperature.

Interfaces at the critical temperature

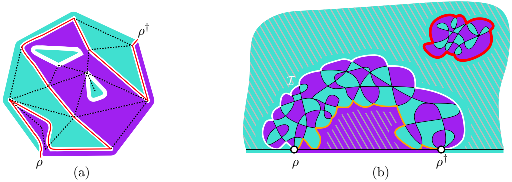

Figure: Spin cluster interfaces when the spins are on faces.

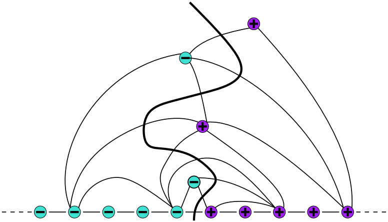

Figure: The unique infinite interface when the spins are on vertices.

A closely related work

- Albenque, Ménard and Schaeffer [2] considered the set of triangulations of the *sphere* of size n decorated with an Ising model on the *vertices*.
- After generalizing [1], they show the local convergence of such triangulations when $n \to \infty$.
- In a recent preprint [3], Albenque and Ménard apply rational parametrizations (including a part of our method) to study the critical perimeter and volume exponents of the spin cluster of the origin.

Works in progress and further directions

- Near-critical regime ($|\nu - \nu_c| \propto p^{-\beta}$)
- Universality (more general lattices)
- More general boundary conditions, yielding to recursion on the generating functions
- Many probabilistic aspects (scaling limits, relations to Liouville Quantum Gravity and Schramm-Loewner Evolutions,...)
- Applications and generalizations of the methods to other statistical mechanics models

This talk was based on:

- L. Chen and J. Turunen. Critical Ising model on random triangulations of the disk: enumeration and local limits. *Commun. Math. Phys.*, 374, 1577–1643 (2020).
- L. Chen and J. Turunen. Ising model on random triangulations of the disk: phase transition. *arXiv:2003.09343*, 2020.

Related works:

- O. Bernardi and M. Bousquet-Melou. Counting colored planar maps: algebraicity results. *J. Combin. Theory Ser. B*, 101(5):315-377, 2011.
- M. Albenque and L. Ménard and G. Schaeffer. Local convergence of large random triangulations coupled with an Ising model. *Trans. Amer. Math. Soc.*, 374(1), 175-217 (2021).
- M. Albenque and L. Ménard. Geometric properties of spin clusters in random triangulations coupled with an Ising model. *arXiv:2201.11922*, 2022.

Merci beaucoup!