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From fighting fish to a formula for Tamari distance

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ongoing work with Enrica Duchi

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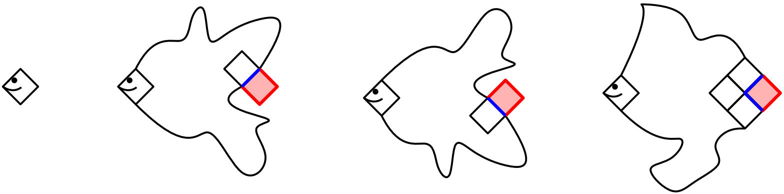
Fighting fish

A **fighting fish** is a set of cells (= unit squares) glued along their edges that can be obtained from the single cell by the application of three rules :

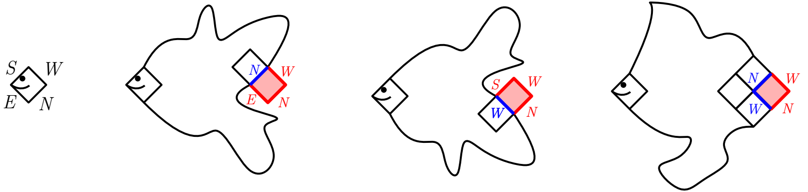


Generalization of parallelogram polyominoes proposed by Duchi et al. (2016).

Fish as words

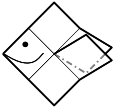
We can also think of fighting fish as words on the alphabet $\{E, N, W, S\}$ (follow its border counterclockwise) generated from the word $ENWS$ by applying the rules :

$$N \to ENW$$
$$W \to NWS$$
$$NW \to WN$$

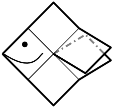


Statistics of fish

We like to present them in the plane even if it is not always possible, and we consider them independantly of the order of construction :



$=$

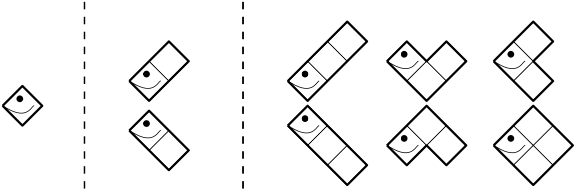


$= \quad E^2N^2WSENW^2S^2$

Size = Halfperimeter - 1
Area = Number of cells
Jaw = Maximal k such that E^k is a prefix

Enumeration

Fighting fish of size ≤ 3 :



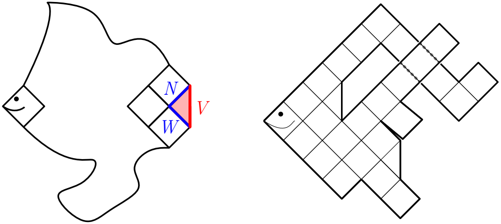
Theorem (Duchi-Rinaldi-Schaeffer-Guerrini '16) :

$$|\mathcal{FF}_n| = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$$

Also counts **synchronized intervals of the Tamari lattice**, nonseparable planar maps, two stack sortable permutations, left ternary trees, ...

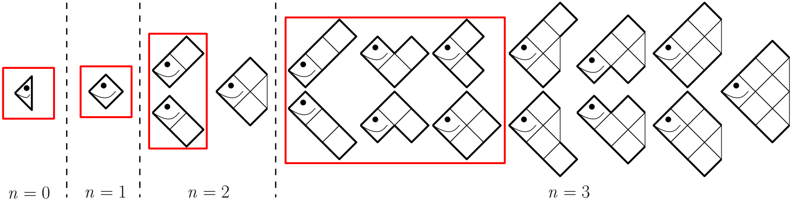
Generalizing fish to fish heads

A **fish head** is a word obtained from *ENWS* using the 3 rules for fighting fish and the additional one $WN \to V$ (triangle gluing) :



Size = Halfperimeter - 1 (not counting V steps)
Area = Number of squares (not counting triangles)
Natural notion of symmetry.

Fish heads of small size



A *i*-pointed fish head is a fish head where a prefix of size $1 \leq i \leq \text{jaw} + 1$ is distinguished. A fighting fish is properly pointed if it is pointed with $i \leq \text{jaw}$.

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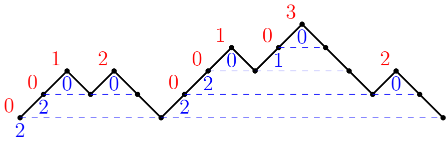
Dyck paths

A **Dyck path of size** n is a path from $(0,0)$ to $(2n,0)$ with n up steps u and n down steps d, staying weakly above the x-axis. Enumeration by Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$.



$\mathbf{D}(P) = (2,3,0,1,0,0,2,1,0,0)$

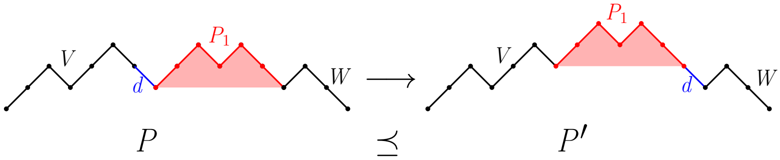
$\mathbf{C}(P) = (2,2,0,0,2,2,0,1,0,0)$

$\mathbf{T}(P) = (1,1,0,0,1,1,0,1,0,0)$

Contact vector $\mathrm{C}(P) = (\mathrm{c}_0(P), \dots, \mathrm{c}_n(P))$, descent vector $\mathrm{D}(P)$, type $\mathrm{T}(P)$, $\mathrm{t}_i(P) = \mathbb{1}_{\mathrm{c}_i(P)\neq 0}$.
The descent vector (or contact vector) fully characterize a Dyck path. Sums to n and verify $\mathrm{d}_0 + \dots + \mathrm{d}_i - i > 0$ for $i \leq n-1$.

The Tamari order on Dyck paths

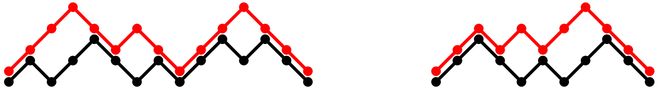
Right rotation on a Dyck path : push a down step d followed by an up step right after the smallest Dyck path following d.
We define $P \preceq P'$ if P' can be obtained from P by a series of right rotations $\rightarrow$ structure of lattice (poset where notions of least supremum, greatest infimum make sense).



A **Tamari interval** is a pair $[P, Q]$ such that $P \preceq Q$. It is **synchronized** if P and Q have the same type.

Examples

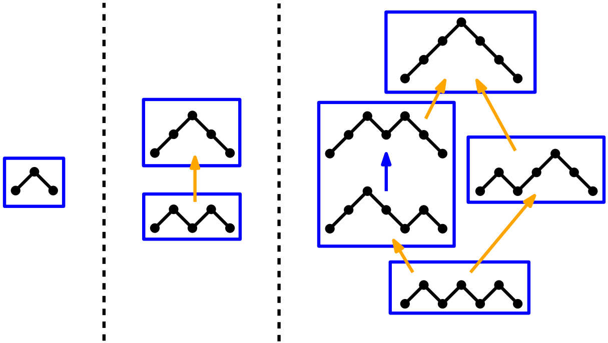
$I = [P, Q] \to \mathrm{C}(I) = \mathrm{C}(P)$, $\mathrm{D}(I) = \mathrm{D}(Q)$



I is i-pointed if distinguished on its i^{th} contact from right to left $(1 \leq i \leq \mathrm{c}_0 + 1)$.
Properly pointed synchronized interval : $i \leq \mathrm{c}_0$.

$n = 1, 2, 3$

In blue boxes, Dyck paths sharing the same type.



A symmetry for Tamari intervals

Conjugation of Dyck paths :

$$\overline{\bullet} = \bullet$$

$$\overline{P_1 u P_2 d} = \overline{P_2} u \overline{P_1} d$$

Involution such that :

$$P \preceq Q \Leftrightarrow \overline{Q} \preceq \overline{P}$$

$$\mathrm{C}(\overline{P}) = \mathrm{D}(P)$$

Natural notion of symmetry for Tamari intervals (and of synchronized intervals).

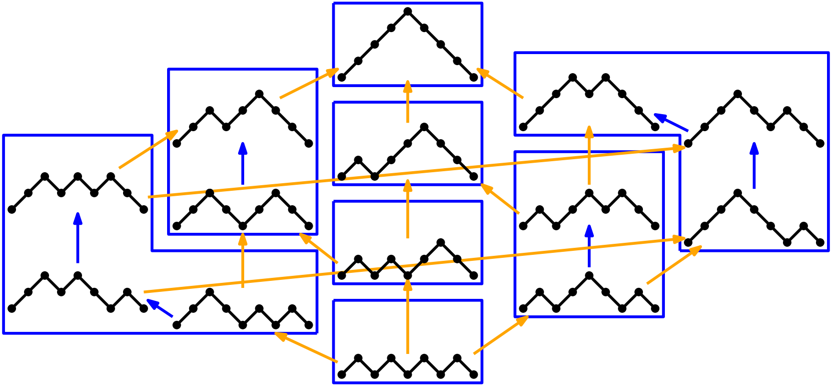
Tamari distance and enumeration of intervals

Tamari distance $\mathbb{d}([P, Q])$: length j of the longest strictly increasing chain $P = P_0 \prec P_2 \prec \ldots \prec P_j = Q$ ($\Rightarrow \mathbb{d}([P, P]) = 0$). Not a distance in the metric sense.
Tamari lattice $(\mathcal{D}_n, \preceq)$ is partitioned into 2^{n-1} sublattices made up of Dyck paths of the same type (Préville-Ratelle, Viennot 2016).

Counting sequences

- $|\mathcal{I}_n| = \frac{2}{n(n+1)}\binom{4n+1}{n-1}$ (Chapoton 2006)
- $|\mathcal{SI}_n| = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$ (Fang, Préville-Ratelle 2016)

$n = 4$



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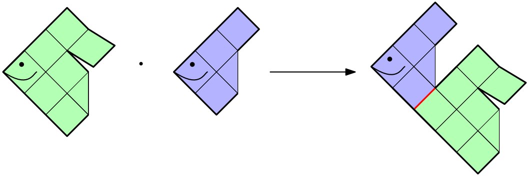
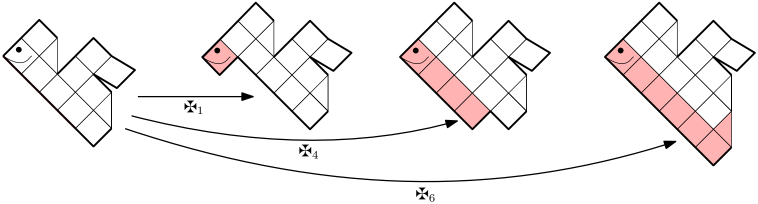
Operations on fish heads

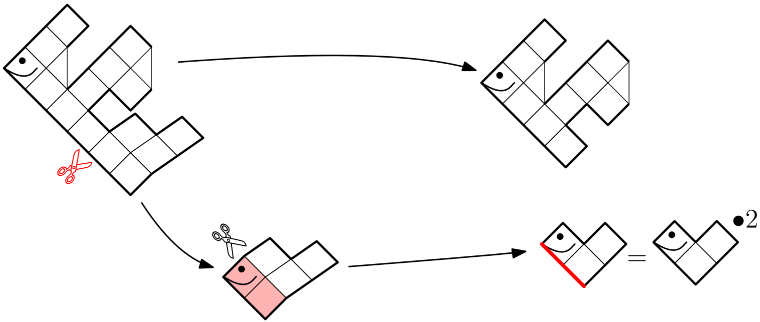
Concatenation of two fish heads :



Augmentation of a i-pointed fish head :



Decomposition of fish heads



Cutting the fish head at its last cut-edge in the jaw yields a decomposition

$$h = \maltese(h_1^{\bullet i}) \cdot h_2$$

Decomposition of fish heads

Theorem

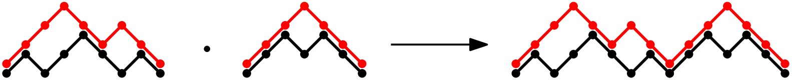
This decomposition is a bijection between $\mathcal{FH} - \{\varepsilon\}$ and $\mathcal{FH}^\bullet \times \mathcal{FH}$ such that if $h = \maltese(h_1^{\bullet i}) \cdot h_2$, we have :

$$\begin{aligned} \mathrm{size}(h) &= \mathrm{size}(h_1) + 1 + \mathrm{size}(h_2) \\ \mathrm{ljaw}(h) &= i + \mathrm{ljaw}(h_2) \\ \mathrm{area}(h) &= \mathrm{area}(h_1) + i + \mathrm{area}(h_2) \end{aligned}$$

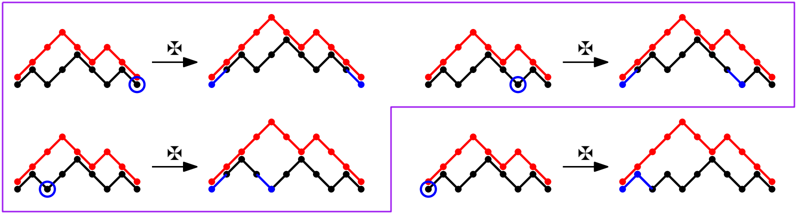
It specializes into a bijection between $\mathcal{FF} - \{\varepsilon\}$ and $\mathcal{FF}^\bullet \times \mathcal{FF}$.

Operations on intervals

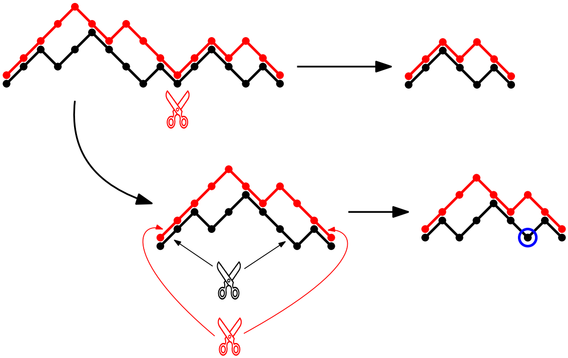
Concatenation of two intervals :



Augmentation of a pointed interval :



Decomposition of Tamari intervals



Cutting the two paths at their first contact yields a decomposition

$$I = \maltese(I_1^{\bullet i}) \cdot I_2$$

Decomposition of Tamari intervals

Theorem (Bousquet-Mélou, Fusy, Préville-Ratelle 2011)

This decomposition is a bijection between $\mathcal{I} - \{[\bullet, \bullet]\}$ and $\mathcal{I}^\bullet \times \mathcal{I}$ such that, if $I = \maltese(I_1^{\bullet i}) \cdot I_2$, we have :

$$\begin{aligned} \mathrm{size}(I) &= \mathrm{size}(I_1) + 1 + \mathrm{size}(I_2) \\ \mathrm{c}_0(I) &= i + \mathrm{c}_0(I_2) \\ \mathbb{d}(I) &= \mathbb{d}(I_1) + i - 1 + \mathbb{d}(I_2) \end{aligned}$$

It specializes into a bijection between $\mathcal{SI} - \{[\bullet, \bullet]\}$ and $\mathcal{SI}^\bullet \times \mathcal{SI}$ (Fang, Préville-Ratelle 2016).

The bijection

Decompositions are isomorphic !

We define recursively :

$$\Phi([\bullet, \bullet]) = \varepsilon$$

$$\Phi(\maltese(I_1^{\bullet i}) \cdot I_2) = \maltese(\Phi(I_1)^{\bullet i}) \cdot \Phi(I_2)$$

Theorem

Φ is a bijection between $\mathcal{I}$ and $\mathcal{FH}$ that specializes into a bijection between $\mathcal{SI}$ and $\mathcal{FH}$. Φ preserves symmetry, and we have :

$$\begin{aligned} \mathrm{size}(I) &= \mathrm{size}(\Phi(I)) \\ \mathrm{c}_0(I) &= \mathrm{jaw}(\Phi(I)) \\ \mathbb{d}(I) &= \mathrm{area}(\Phi(I)) - \mathrm{size}(\Phi(I)) \end{aligned}$$

A direct description of Φ

Computations allowed by this recursive bijection gave us insights about a direct description of Φ :

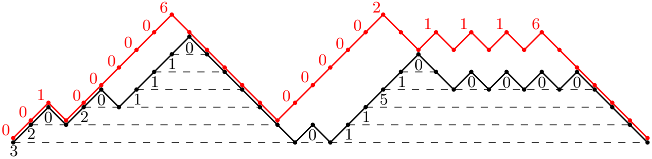
Theorem

Let $[P, Q]$ be a Tamari interval. For each $0 \leq i \leq n$, we define the word w_i by :

$$
\begin{aligned}
w_i &= E^{\mathrm{c}_i(P)-1}N & &\text{if } \mathrm{c}_i(P) \geq 1 \text{ and } \mathrm{d}_{n-i}(Q) = 0 \\
w_i &= WS^{\mathrm{d}_{n-i}(Q)-1} & &\text{if } \mathrm{c}_i(P) = 0 \text{ and } \mathrm{d}_{n-i}(Q) \geq 1 \\
w_i &= V & &\text{if } \mathrm{c}_i(P) = 0 \text{ and } \mathrm{d}_{n-i}(Q) = 0
\end{aligned}
$$

Then $w = Ew_0w_1...w_nS$ is a fish head and $w = \Phi([P, Q])$

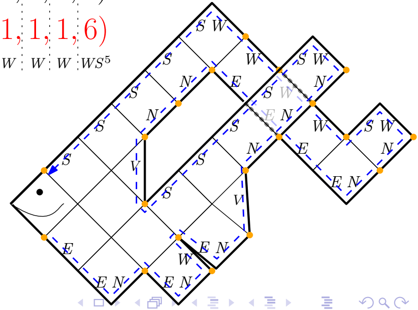
Working an example



$$\mathbf{C}(I) = (3,2,0,2,0,1,1,1,0,0,1,1,5,1,0,0,0,0,0)$$

$$\overleftarrow{\mathbf{D}}(I) = (0,0,1,0,0,0,0,0,6,0,0,0,0,0,2,1,1,1,6)$$

E^2N EN W EN V N N N WS^5 V N N E^4N N WS W W W WS^5



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Motivation

Tamari distance extends to intervals the notion of area of a Dyck path :

$$\mathbb{d}([udud...ud, P]) = \text{area}(P)$$

It seems to be an important statistic for understanding combinatorics of diagonal harmonics (theory of representation of the symmetric group, extensions of the shuffle conjecture) :

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Such a statistic would give a complete combinatorial description of $\mathcal{H}_n^{(r)}(\mathbf{w}; q_1, q_2, q_3)$ in the form

$$\mathcal{H}_n^{(r)}(\mathbf{w}; q_1, q_2, q_3) = \sum_{\beta \in \mathcal{D}_n^{(r)}} \sum_{f \in \mathcal{PF}(\beta)} \sum_{\alpha \leq \beta} q_1^{d(\alpha,\beta)} q_2^{\mathrm{dinv}(f)} q_3^{\nu(f,\alpha)} Q_{\mathbf{co}(f)}(\mathbf{w}). \qquad (49)$$

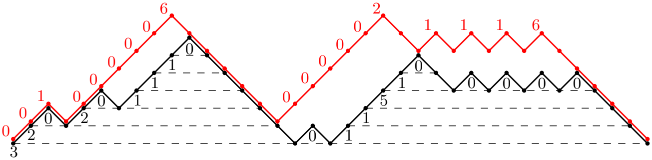
How fish heads help

Two points of view on fish heads :

- gluing of cells
- counterclockwise tour of the border

The second one is natural for the bijection with Tamari intervals and allows us to compute the area of the fish head with the corresponding contact and descent vectors.

On the example

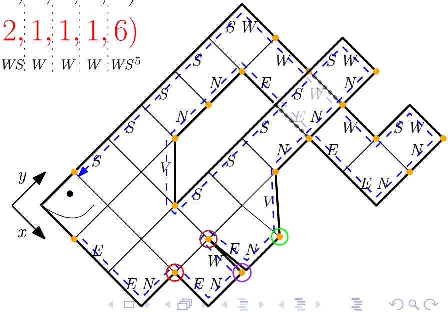


$$\mathbf{C}(I) = (3, 2, 0, 2, 0, 1, 1, 1, 0, 0, 1, 1, 5, 1, 0, 0, 0, 0, 0)$$

$$\overleftarrow{\mathbf{D}}(I) = (0, 0, 1, 0, 0, 0, 0, 0, 6, 0, 0, 0, 0, 0, 2, 1, 1, 1, 6)$$

E^2N EN W EN V N N N WS^5 V N N E^4N N WS W W W WS^5

- $(\mathbf{c}_0, 1 - \mathbf{d}_n)$
- $(\mathbf{c}_0 + \mathbf{c}_1 - 1, 2 - \mathbf{d}_n - \mathbf{d}_{n-1})$
- $(\mathbf{c}_0 + \mathbf{c}_1 + \mathbf{c}_3 - 2, 3 - \mathbf{d}_n - \mathbf{d}_{n-1} - \mathbf{d}_{n-2})$
-



A formula for the Tamari distance

Theorem

For every interval $[P, Q]$ in the Tamari lattice :

$$\mathbb{d}([P, Q]) = \sum_{0 \leq i < j \leq n} (\mathrm{c}_i(P) - 1)(1 - \mathrm{d}_{n-j}(Q))$$

The duality relation $\mathbb{d}([\overline{Q}, \overline{P}]) = \mathbb{d}([P, Q])$ can be directly seen on the formula.

Easy to compute.

What does it mean for $P \nleq Q$?

Applications

The average area of uniform fighting fish grows like $n^{\frac{5}{4}}$ (Duchi et al. 2016) and so do the average Tamari distance of uniform synchronized intervals.

Extension to m-Tamari lattice (an extension to m-Dyck paths) possible using the inclusion of m-Tamari in Tamari.

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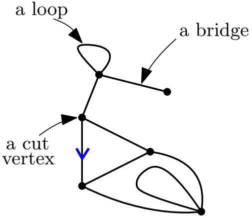
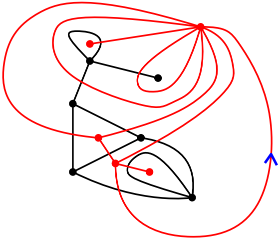
Rooted planar maps

A **planar map** is a proper embedding of a connected graph on the plane, defined up to continuous deformations. We consider **rooted** planar maps : we distinguish and orient an edge in such a way that the outer face is on its right.
Size = number of edges A natural notion of symmetry is **duality**.

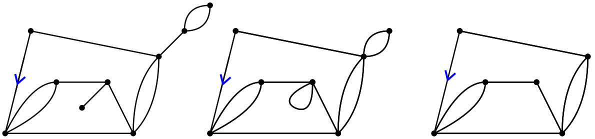




Bridgeless, loopless, non-separable maps

A rooted planar map is :

- **bridgeless** if it contains no bridge
- **loopless** if it contains no loop
- **non-separable** if it contains no cut vertex

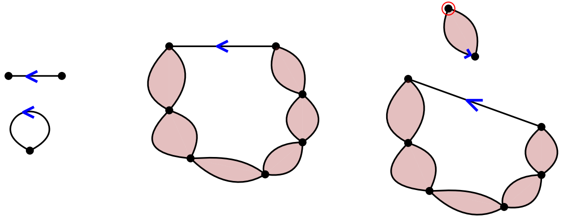


Enumerating sequences

$$|\mathcal{BPM}_n| = |\mathcal{LPM}_n| = |\mathcal{I}_n| = |\mathcal{FH}_n|$$
$$|\mathcal{NSPM}_{n+1}| = |\mathcal{SI}_n| = |\mathcal{FF}_n|$$

An isomorphic decomposition for $\mathcal{NSPM}$

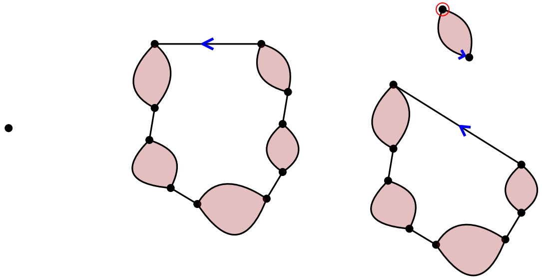
We can decompose a non-separable planar map by cutting it at its last cut vertex obtained if we delete the root edge.
Pointing : on a non-root vertex of the outer face



Isomorphic decomposition as for fighting fish and synchronized intervals (also preserves symmetry)

An isomorphic decomposition for $\mathcal{BPM}$

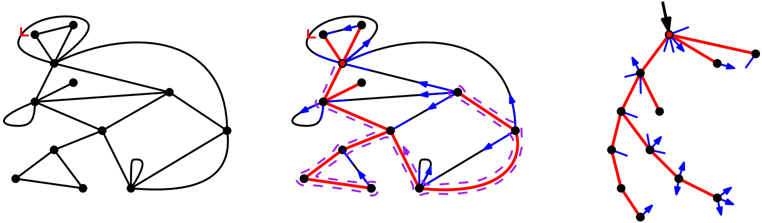
We can decompose a bridgeless planar map by cutting it at its last bridge obtained if we delete the root edge.
Pointing : on a (possibly) root vertex of the outer face



Isomorphic decomposition as for fish heads and Tamari intervals (do not preserve symmetry)

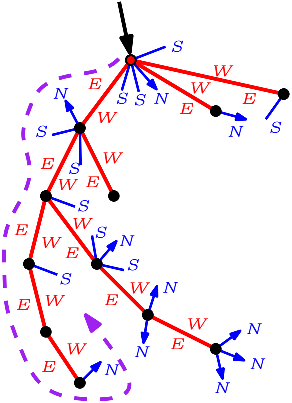
A description with decorated trees

We endow a rooted planar map with its unique rightmost depth first search spanning tree starting from the edge following the root edge, add an opening half-edge each time we encounter a never visited edge going to a previously visited vertex and add a closing half-edge the second time we encounter such an edge. We call this algorithm the **counterclockwise tour** of the map.



From decorated trees to $\{E, N, W, S\}$ words

We obtain a word on the alphabet $\{E, N, W, S\}$ when we follow edges of the decorated tree : this is the Mullin-Schellenberg encoding of a rooted planar map. Non-separable maps $\rightarrow$ fighting fish. Generalized fighting fish for general rooted maps do not allow to extend the gluing of cells approach.



Some questions

What does the area of fighting fish mean on NSP maps ?

How to extend fish heads to maps keeping a natural notion of symmetry ?

Extension of the fish/map framework to intervals of the m-Tamari lattice ?

Understand the link between synchronized intervals and two stack permutations.

Thank you for your attention !