Cylindric partitions and Rogers–Ramanujan type identities

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Outline

1. Integer partitions, Rogers–Ramanujan type identities and a bit of representation theory
2. Cylindric partitions
3. q-difference equations from cylindric partitions and Rogers-Ramanujan type identities

Integer partitions and compositions

Definition

A *partition* λ of a positive integer n is a finite non-increasing sequence of positive integers $(\lambda_1, \ldots, \lambda_m)$ such that $\lambda_1 + \cdots + \lambda_m = n$. The integers $\lambda_1, \ldots, \lambda_m$ are called the *parts* of the partition λ.

Example

There are 5 partitions of 4: $4, (3,1), (2,2), (2,1,1)$ and $(1,1,1,1)$.

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Example

There are 5 partitions of 4: $4, (3,1), (2,2), (2,1,1)$ and $(1,1,1,1)$.

Definition

A *composition* c of a positive integer n is a finite sequence (with no restriction on the order) of positive integers $(c_1, \ldots, c_m)$ such that $c_1 + \cdots + c_m = n$.

Example

$(1,2,1)$ is a composition of 4 but not a partition.

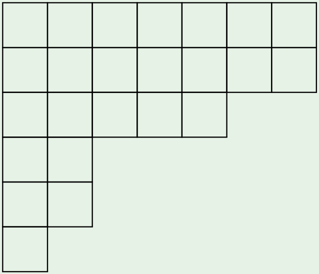
Young diagrams

Definition

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition. The *Young diagram* of λ is a finite collection of boxes arranged in left-justified rows, with λ_i boxes in the i-th row for all $1 \leq i \leq m$.

Example

$\lambda = (7, 7, 5, 2, 2, 1)$ is a partition of 24 with Young diagram



Generating functions

Notation : $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), n \in \mathbb{N} \cup \{\infty\}$.

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Let $Q(n,k)$ be the number of partitions of n into k distinct parts. Then

$$\begin{aligned} 1 + \sum_{n\geq 1}\sum_{k\geq 1} Q(n,k) z^k q^n &= (1+zq)(1+zq^2)(1+zq^3)(1+zq^4)\cdots \\ &= (-zq;q)_\infty. \end{aligned}$$

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Let $p(n,k)$ be the number of partitions of n into k parts. Then

$$\begin{aligned} 1 + \sum_{n\geq 1}\sum_{k\geq 1} p(n,k)z^k q^n &= \prod_{n\geq 1}\left(1+zq^n+z^2q^{2n}+\cdots\right) \\ &= \frac{1}{(zq;q)_\infty}. \end{aligned}$$

Generating functions

More generally:

- The generating function for partitions into distinct parts congruent to $k \mod N$ is
$$(-zq^k; q^N)_\infty.$$
- The generating function for partitions into parts congruent to $k \mod N$ is
$$\frac{1}{(zq^k; q^N)_\infty}.$$

q-binomial coefficients

Definition

For two integers n and m, the q-binomial coefficient $\begin{bmatrix} m+n \\ m \end{bmatrix}_q$ is defined as

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_{m+n}}{(q;q)_m(q;q)_n} & \text{for } m, n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is the generating function for partitions whose Young diagram fits inside an $n \times m$ rectangle.

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Properties

- ${m+n \brack m}_q = {m+n \brack n}_q$
- $\lim_{n\to\infty} {m+n \brack m}_q = 1/(q;q)_m$

The first Rogers–Ramanujan identity

Theorem (Rogers 1894, Rogers–Ramanujan 1919)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

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Theorem (Partition version)

For every positive integer n, the number of partitions of n such that the difference between two consecutive parts is at least 2 is equal to the number of partitions of n into parts congruent to 1 or 4 modulo 5.

Some definitions on Lie algebras

Definition

A *Lie algebra* $\mathfrak{g}$ is a vector space together with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket, satisfying:

- alternativity : for all $x \in \mathfrak{g}$, $[x,x] = 0$,
- the Jacobi identity: for all $x, y, z \in \mathfrak{g}$, $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$.

Example

The *special linear Lie algebra* of order n, denoted A_{n-1} or $\mathfrak{sl}_n(\mathbb{C})$, is the Lie algebra of $n \times n$ matrices with trace zero and with the Lie bracket $[X,Y] = XY - YX$.

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Definition

A *representation (or module)* of $\mathfrak{g}$ is a vector space V together with a linear map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, such that $\rho([X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$.

Some definitions on Lie algebras

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra with Cartan subalgebra $\mathfrak{h}$. The corresponding (derived) affine Lie algebra $\hat{\mathfrak{g}}$ is constructed as

$$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where $\mathbb{C}[t, t^{-1}]$ is the complex vector space of Laurent polynomials in the indeterminate t, and $\mathbb{C}c$ is $\hat{\mathfrak{g}}$'s center (one-dimensional).

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If V is an irreducible highest weight module of $\hat{\mathfrak{g}}$, the central element c acts on V by multiplication by a scalar k, which is called the **level** of V. The **character** $\mathrm{ch}(V)$ of $V = \bigoplus_\mu V_\mu$ is defined as

$$\mathrm{ch}(V) = \sum_\mu \dim(V_\mu) e^\mu,$$

where the sum is over the weights μ of V,
$V_\mu := \{v \in V : \forall H \in \mathfrak{h}, \quad H \cdot v = \mu(H)v\}$ is a weight space,
and e^μ is a formal exponential satisfying $e^\mu e^{\mu'} = e^{\mu+\mu'}$.

Representation theoretic interpretation

Lepowsky and Wilson 1984: representation theoretic interpretation

$$\frac{1}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q;q^2)_\infty}\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}$$

Obtained by giving two different formulations for the principal specialisation

$$e^{-\alpha_0}\mapsto q,\quad e^{-\alpha_1}\mapsto q$$

of $e^{-\Lambda}\mathrm{ch}(L(\Lambda))$ where $L(\Lambda)$ is an irreducible highest weight $A_1^{(1)}$-module of level 3.

RHS: principal specialisation of the Weyl-Kac character formula

LHS: comes from the construction of a basis of $L(\Lambda)$ using vertex operators

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LHS: comes from the construction of a basis of $L(\Lambda)$ using vertex operators.

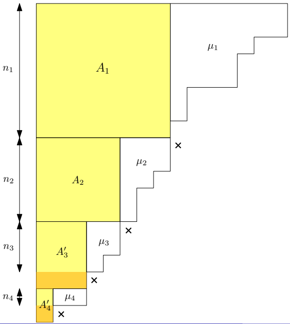
Very rough idea:

- Start with a spanning set of $L(\Lambda)$: here, monomials of the form $Z_1^{f_1}\dots Z_s^{f_s}$ for $s, f_1, \dots, f_s \in \mathbb{N}_{\geq 0}$.
- Using Lie theory, reduce this spanning set: here, it allows one to remove all monomials containing Z_j^2 or Z_jZ_{j+1}.
- Show that the obtained set is a basis of the representation (very difficult).

The Andrews–Gordon identities

Let $r \geq 2$ and $1 \leq i \leq r$ be two integers. We have

$$\sum_{n_1 \geq \cdots \geq n_{r-1} \geq 0} \frac{q^{n_1^2+\cdots+n_{r-1}^2+n_i+\cdots+n_{r-1}}}{(q;q)_{n_1}} \begin{bmatrix} n_1 \\ n_1-n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{r-2} \\ n_{r-2}-n_{r-1} \end{bmatrix}_q$$
$$= \frac{(q^{2r+1}, q^i, q^{2r-i+1}; q^{2r+1})_\infty}{(q;q)_\infty}.$$



Corresponds to characters of higher level modules of $A_1^{(1)}$ (Meurman–Primc 1987)

Interactions between combinatorics and representation theory

From the combinatorial point of view: representation theory is a great source for conjecturing new partition identities:

Capparelli 1993: level 3 standard modules of $A_2^{(2)}$

Nandi 2014: level 4 standard modules of $A_2^{(2)}$

Siladić 2002: twisted level 1 modules of $A_2^{(2)}$

Primc 1999: $A_2^{(1)}$ and $A_1^{(1)}$ crystals

Primc and Šikić 2016: level k standard modules of $C_n^{(1)}$

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From the representation theoretic point of view: combinatorics can help finding expressions of the character $\mathrm{ch}(V) = \sum_\mu \dim(V_\mu) e^\mu$ as a series with obviously positive coefficients.

Andrews–Schilling–Warnaar 1999, Bartlett–Warnaar 2015, D.–Konan 2020, ...

Cylindric partitions can be used to do so (last section).

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Plane partitions

Definition

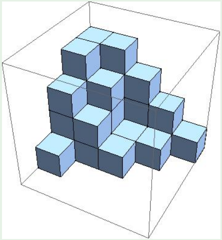
A *plane partition* is a vector partition $\Lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)})$, where each $\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_{s_i}^{(i)})$ is a partition, such that for all i and j,

$$\lambda_j^{(i)} \geq \lambda_j^{(i+1)}.$$

Example

$((4,4,3,2,1),(4,3,1,1),(3,2,1),(1))$

4	4	3	2	1
4	3	1	1	
3	2	1		
1				



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$$\lambda_j^{(i)} \geq \lambda_j^{(i+1)}.$$

The *sum* of Λ is the sum of all the parts in the partitions $\lambda^{(1)}, \dots, \lambda^{(k)}$. Let $PL(n)$ denote the number of plane partitions with sum n.

Theorem (MacMahon 1916)

The generating function for plane partitions is

$$\sum_{n \geq 0} PL(n) q^n = \prod_{i \geq 1} \frac{1}{(1-q^i)^i} \cdot$$

Cylindric partitions

Definition (Gessel–Krattenthaler 1997)

Let $c = (c_1, \dots, c_k)$ be a composition.
A *cylindric partition with profile c* is a vector partition $\Lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)})$, where each $\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_{s_i}^{(i)})$ is a partition, such that for all i and j,

$$\lambda_j^{(i)} \geq \lambda_{j+c_{i+1}}^{(i+1)} \quad \text{and} \quad \lambda_j^{(k)} \geq \lambda_{j+c_1}^{(1)}.$$

The *sum* $|\Lambda|$ of the cylindric partition Λ is the sum of all the parts in the partitions $\lambda^{(1)}, \dots, \lambda^{(k)}$.
Its largest part $\max(\Lambda)$ is the largest part among all the partitions $\lambda^{(1)}, \dots, \lambda^{(k)}$.
Let $\mathcal{P}_c$ denote the set of cylindric partitions with profile c.

Cylindric partitions: example

Consider the composition $c = (3, 1, 1)$.

A cylindric partition with profile c is a vector partition $\Lambda = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$, where each $\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_{s_i}^{(i)})$ is a partition, such that for all j,

$$\lambda_j^{(1)} \geq \lambda_{j+1}^{(2)}, \quad \lambda_j^{(2)} \geq \lambda_{j+1}^{(3)}, \quad \text{and} \quad \lambda_j^{(3)} \geq \lambda_{j+3}^{(1)}.$$

For example, $\Lambda = ((4, 4, 3, 2, 2, 1), (5, 4, 3, 2, 2), (3, 2, 1, 1))$ works.

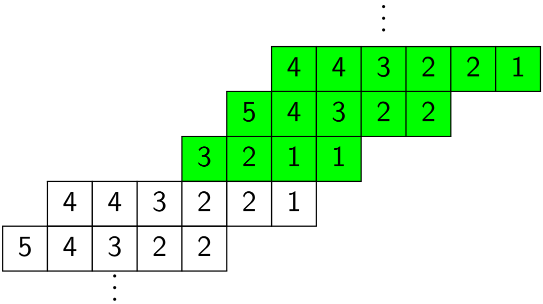
								$\vdots$			
						4	4	3	2	2	1
					5	4	3	2	2		
				3	2	1	1				
	4	4	3	2	2	1					
5	4	3	2	2							
		$\vdots$									

A simple bijection

Proposition

For any composition $c = (c_1, \ldots, c_{k-1}, c_k)$, the set of cylindric partitions with profile c is in bijection the set of cylindric partitions with profile $c' = (c_k, c_1, \ldots, c_{k-1})$.

$c = (3, 1, 1)$:



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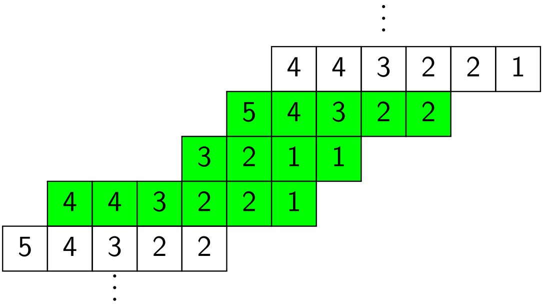
$c = (1, 3, 1)$:



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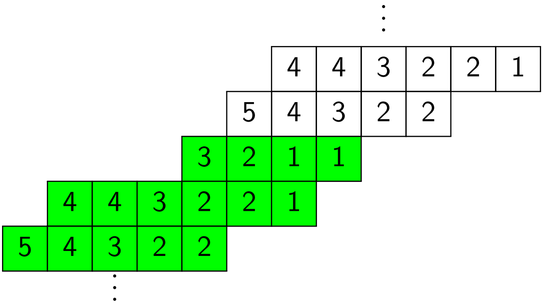
$c = (1, 1, 3)$:



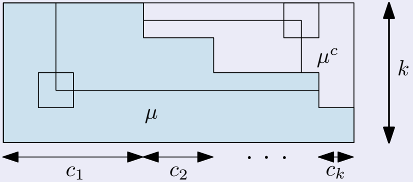
Generating function

Theorem (Borodin 2007)

Let k and ℓ be positive integers, and let $c = (c_1, c_2, \ldots, c_k)$ be a composition of ℓ. Define $t := k + \ell$.
Let $F_c(z, q) := \sum_{\Lambda \in \mathcal{P}_c} z^{\max(\Lambda)} q^{|\Lambda|}$ be the generating function for cylindric partitions with profile c. We have:

$$F_c(1, q) = \frac{1}{(q^t; q^t)_\infty} \prod_{\square \in \mu} \frac{1}{(q^{h(\square)}; q^t)_\infty} \prod_{\square \in \mu^c} \frac{1}{(q^{t-h(\square)}; q^t)_\infty},$$

where μ is the partition $(c_1 + \cdots + c_k, \ldots, c_2 + c_1, c_1)$ and $h(\square)$ denotes the hook length of the box $\square$:



Generating function for cylindric partitions with profile $c = (3, 1, 1)$

$$F_c(1, q) = \frac{1}{(q^t; q^t)_\infty} \prod_{\square \in \mu} \frac{1}{(q^{h(\square)}; q^t)_\infty} \prod_{\square \in \mu^c} \frac{1}{(q^{t-h(\square)}; q^t)_\infty},$$

3	2	1	7	5
5	4	3	1	7
7	6	5	3	1

We have $t = 5 + 3 = 8$, and the generating function for cylindric partitions with profile $(3, 1, 1)$ is

$$F_{(3,1,1)}(1, q) = \frac{1}{(q; q)_\infty} \times \frac{1}{(q, q, q^3, q^3, q^5, q^5, q^7, q^7; q^8)_\infty}.$$

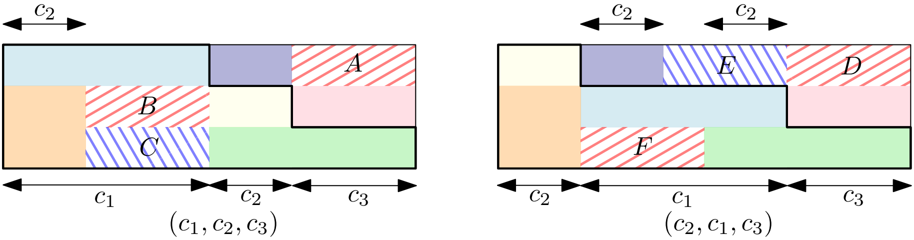
A new symmetry

Theorem (Corteel–D.–Uncu 2020)

For any non-negative integers c_1, c_2, c_3, we have

$$F_{(c_1,c_2,c_3)}(1,q) = F_{(c_2,c_1,c_3)}(1,q).$$

Proof:



$$\{h(\square) | \square \in C\} = \{t - h(\square) | \square \in E\} = \{c_2 + c_3 + 3, \dots, c_1 + c_3 + 2\},$$
$$\{h(\square) | \square \in B\} \cup \{t - h(\square) | \square \in A\} = \{h(\square) | \square \in F\} \cup \{t - h(\square) | \square \in D\}.$$

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A general q-difference equation for cylindric partitions

Theorem (Corteel-Welsh 2019)

Let $c = (c_1, \ldots, c_k)$ *and use the convention that* $c_0 = c_k$. *Denote by* I_c *the set of indices* $j \in \{1, \ldots, k\}$ *such that* $c_j > 0$. *Given a subset* J *of* I_c, *the composition* $c(J) = (c_1(J), \ldots, c_k(J))$ *is defined by:*

$$c_i(J) := \begin{cases} c_i - 1 & \text{if } i \in J \text{ and } i-1 \notin J, \\ c_i + 1 & \text{if } i \notin J \text{ and } i-1 \in J, \\ c_i & \text{otherwise.} \end{cases}$$

Then

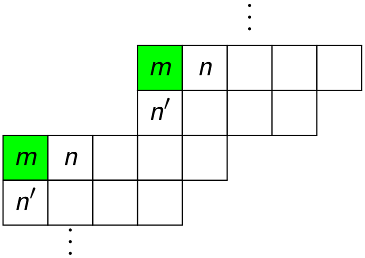
$$F_c(z,q) = \sum_{\emptyset \subset J \subseteq I_c} (-1)^{|J|-1} \frac{F_{c(J)}(zq^{|J|}, q)}{(1 - zq^{|J|})},$$

with the initial conditions $F_c(0,q) = F_c(z,0) = 1$.

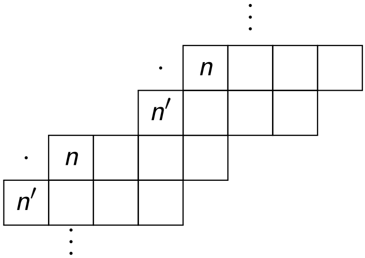
Example of profiles $(3,0)$ and $(2,1)$

$$F_{(3,0)}(z,q) = \frac{1}{1-zq} F_{(2,1)}(zq,q)$$

Proof:



$(3,0)$



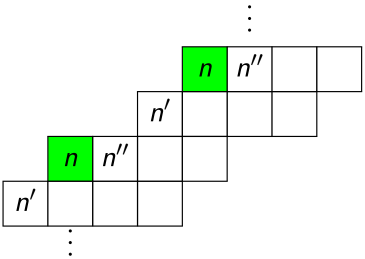
$(2,1)$

$$m = \max(n, n') + k, \ \ k \in \mathbb{N}.$$

Example of profiles $(3,0)$ and $(2,1)$

$$F_{(2,1)}(z,q) = \frac{1}{1-zq}F_{(2,1)}(zq,q) + \frac{1}{1-zq}F_{(3,0)}(zq,q) - \frac{1}{1-zq^2}F_{(2,1)}(zq^2,q)$$

Proof: Case 1: $n \geq n'$



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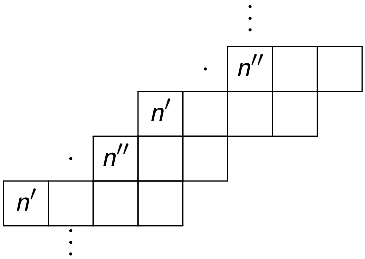

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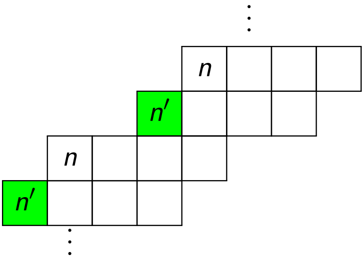
Proof: Case 2: $n' \geq n$



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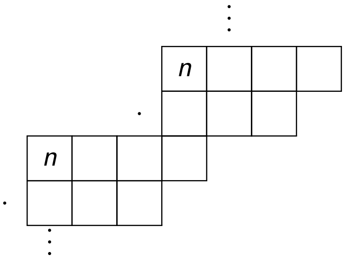


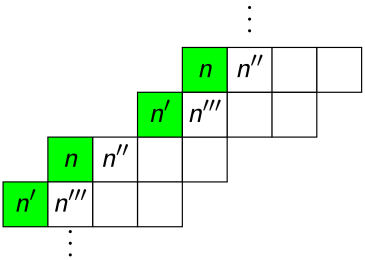
$(3,0)$

$$n' = n + k, \;\; k \in \mathbb{N}.$$

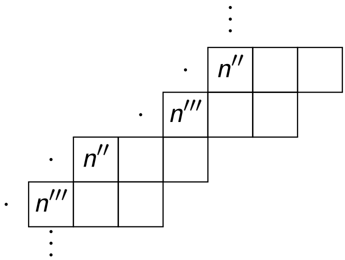
Example of profiles $(3,0)$ and $(2,1)$

$$F_{(2,1)}(z,q) = \frac{1}{1-zq}F_{(2,1)}(zq,q) + \frac{1}{1-zq}F_{(3,0)}(zq,q) - \frac{1}{1-zq^2}F_{(2,1)}(zq^2,q)$$

Proof: Case 3: $n' = n$ (already counted twice!)



$(2,1)$



$(2,1)$

$$n' = n = \max(n'', n''') + k, \ \ k \in \mathbb{N}.$$

Reproving the Rogers–Ramanujan identities

For any composition c, let

$$G_c(z,q) := (zq;q)_\infty F_c(z,q).$$

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Let us solve this system of q-difference equations!

Reproving the Rogers–Ramanujan identities

$$G_{(3,0)}(z,q) = G_{(2,1)}(zq,q), \quad (1)$$

$$G_{(2,1)}(z,q) = G_{(3,0)}(zq,q) + G_{(2,1)}(zq,q) - (1-zq)G_{(2,1)}(zq^2,q), \quad (2)$$

with the initial conditions $G_c(0,q) = G_c(z,0) = 1$.

Substituting (1) into (2), we obtain

$$G_{(2,1)}(z,q) = G_{(2,1)}(zq,q) + zqG_{(2,1)}(zq^2,q).$$

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Writing $G_{(2,1)}(z,q) = \sum_{n\geq 0} a_n(q) z^n$, then $a_0(q) = 1$ and

$$a_n = a_n(q)q^n + a_{n-1}(q)q^{2n-1}.$$

Iterating, we obtain

$$a_n(q) = \frac{q^{2n-1}}{1-q^n} a_{n-1}(q) = \frac{q^{n^2}}{(q;q)_n}.$$

Reproving the Rogers–Ramanujan identities

We obtain

$$G_{(2,1)}(z,q) = \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} z^n,$$

and by (1),

$$G_{(3,0)}(z,q) = \sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n} z^n.$$

Using the product formulas of Borodin, we recover the two Rogers–Ramanujan identities:

$$G_{(2,1)}(1,q) = \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty},$$

$$G_{(3,0)}(1,q) = \sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}.$$

Cylindric partitions and the Andrews–Gordon identities

Let $r \geq 2$ and $1 \leq i \leq r$ be two integers. We have

$$\sum_{n_1 \geq \cdots \geq n_{r-1} \geq 0} \frac{q^{n_1^2+\cdots+n_{r-1}^2+n_i+\cdots+n_{r-1}}}{(q;q)_{n_1}} \begin{bmatrix} n_1 \\ n_1-n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{r-2} \\ n_{r-2}-n_{r-1} \end{bmatrix}_q$$
$$= \frac{(q^{2r+1}, q^i, q^{2r-i+1}; q^{2r+1})_\infty}{(q;q)_\infty}.$$

Using cylindric partitions with profile $c = (2r-i, i-1)$ and certain types of lattice paths, Foda and Welsh (2016) reproved this identity.

Back to representation theory

We saw that the Rogers–Ramanujan and Andrews–Gordon identities are related to characters of the Lie algebra A_1.

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The W_n algebra is an A_{n-1} generalisation of the famous Virasoro algebra.

In 1999, Andrews, Schilling and Warnaar developed a so-called "A_2 Bailey lemma" and used it to study an infinite family of characters of W_3. A particular case is the following:

$$\mathrm{ch} = (q,q)_\infty \sum_{a_1,b_1,a_2,b_2 \in \mathbb{Z}} \frac{q^{a_1^2+b_1^2+a_2^2+b_2^2-a_1b_1+a_2b_2+a_1+a_2+b_1+b_2}}{(q;q)_{a_1-a_2}(q;q)_{b_1-b_2}(q;q)_{a_2}(q;q)_{b_2}(q;q)_{a_2+b_2+1}}$$

$$= \frac{1}{(q^2,q^3,q^3,q^4,q^4,q^5,q^5,q^6;q^8)_\infty}$$

Problem: the sum does not have obviously positive coefficients.

Back to representation theory

In four particular cases, Andrews, Schilling and Warnaar were able to rewrite the sum so that it has obviously positive coefficients. The corresponding identities are called A_2 **Rogers–Ramanujan identities**.

An A_2 Rogers–Ramanujan identity (Andrews–Schilling–Warnaar 1999)

$$\sum_{n_1,n_2\geq 0} \frac{q^{n_1^2+n_2^2-n_1n_2}}{(q;q)_{n_1}} \begin{bmatrix} 2n_1 \\ n_2 \end{bmatrix}_q = \frac{1}{(q,q,q^3,q^4,q^6,q^6;q^7)_\infty}.$$

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In 2019, Corteel and Welsh reproved the four identities of Andrews–Schilling–Warnaar; together with a fifth one, by studying all cylindric partitions with profiles $c = (c_1, c_2, c_3)$ with $c_1 + c_2 + c_3 = 4$.

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Natural next step: profiles $c=(c_1,c_2,c_3)$ with $c_1+c_2+c_3=5$.

Cylindric partitions with profiles $c = (c_1, c_2, c_3)$ with $c_1 + c_2 + c_3 = 5$

By Borodin's theorem and the symmetries previously mentioned, the *exhaustive* list of generating functions for partitions with profiles $c = (c_1, c_2, c_3)$ with $c_1 + c_2 + c_3 = 5$ is:

$$\begin{aligned}
G_{(5,0,0)}(1,q) &= \frac{1}{(q^2,q^3,q^3,q^4,q^4,q^5,q^5,q^6;q^8)_\infty},\\
G_{(4,1,0)}(1,q) = G_{(4,0,1)}(1,q) &= \frac{1}{(q,q^2,q^3,q^4,q^4,q^5,q^6,q^7;q^8)_\infty},\\
G_{(3,0,2)}(1,q) = G_{(3,2,0)}(1,q) &= \frac{1}{(q,q^2,q^2,q^3,q^5,q^6,q^6,q^7;q^8)_\infty},\\
G_{(3,1,1)}(1,q) &= \frac{1}{(q,q,q^3,q^3,q^5,q^5,q^7,q^7;q^8)_\infty},\\
G_{(2,2,1)}(1,q) &= \frac{1}{(q,q,q^2,q^4,q^4,q^6,q^7,q^7;q^8)_\infty}.
\end{aligned}$$

Cylindric partitions with profiles $c = (c_1, c_2, c_3)$ with $c_1 + c_2 + c_3 = 5$

By Corteel-Welsh, we obtain a system a q-difference equations for these generating functions:

$$\begin{aligned}
G_{(5,0,0)}(z,q) &= G_{(4,1,0)}(zq,q),\\
G_{(4,1,0)}(z,q) &= G_{(4,0,1)}(zq,q) + G_{(3,2,0)}(zq,q) - (1-zq)G_{(3,1,1)}(zq^2,q),\\
G_{(4,0,1)}(z,q) &= G_{(5,0,0)}(zq,q) + G_{(3,1,1)}(zq,q) - (1-zq)G_{(4,1,0)}(zq^2,q),\\
G_{(3,2,0)}(z,q) &= G_{(3,1,1)}(zq,q) + G_{(3,0,2)}(zq,q) - (1-zq)G_{(2,2,1)}(zq^2,q),\\
G_{(3,1,1)}(z,q) &= G_{(4,1,0)}(zq,q) + G_{(3,0,2)}(zq,q) + G_{(2,2,1)}(zq,q)\\
&\quad - (1-zq)\big(G_{(4,0,1)}(zq^2,q) + G_{(3,2,0)}(zq^2,q) + G_{(2,2,1)}(zq^2,q)\big)\\
&\quad + (1-zq)(1-zq^2)G_{(3,1,1)}(zq^3,q),\\
G_{(3,0,2)}(z,q) &= G_{(4,0,1)}(zq,q) + G_{(2,2,1)}(zq,q) - (1-zq)G_{(3,1,1)}(zq^2,q),\\
G_{(2,2,1)}(z,q) &= G_{(3,2,0)}(zq,q) + G_{(3,1,1)}(zq,q) + G_{(2,2,1)}(zq,q)\\
&\quad - (1-zq)\big(G_{(3,1,1)}(zq^2,q) + G_{(3,0,2)}(zq^2,q) + G_{(2,2,1)}(zq^2,q)\big)\\
&\quad + (1-zq)(1-zq^2)G_{(2,2,1)}(zq^3,q).
\end{aligned}$$

Cylindric partitions with profiles $c = (c_1, c_2, c_3)$ with $c_1 + c_2 + c_3 = 5$

Writing, for all c,

$$G_c(z, q) = \sum_{k \geq 0} g_c(k) z^k,$$

we transform these q-difference equations into a system of recurrences on the $(g_c(k))$. For example:

$$g_{(4,1,0)}(k) = q^k g_{(4,0,1)}(k) + q^k g_{(3,2,0)}(k) - q^{2k} g_{(3,1,1)}(k) + q^{2k-1} g_{(3,1,1)}(k-1).$$

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Then we use Gröbner bases calculations, performed automatically in the `HolonomicFunctions` package (Koutschan 2009), to uncouple this system of recurrences and obtain a single (much longer) recurrence satisfied by each of the sequences $(g_c(k))$.

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These equations are hard to solve, but we had **conjectures** for the solutions.

Cylindric partitions with profiles $c = (c_1, c_2, c_3)$ with $c_1 + c_2 + c_3 = 5$

Example of conjecture:

$$g_{(4,1,0)}(k) = \sum_{n_2,n_3,n_4 \geq 0} \frac{q^{k^2+n_2^2+n_3^2+n_4^2+n_2+n_3+n_4-kn_2+n_2n_4}}{(q;q)_k} \begin{bmatrix} k \\ n_2 \end{bmatrix}_q \begin{bmatrix} k \\ n_4 \end{bmatrix}_q \begin{bmatrix} n_2 \\ n_3 \end{bmatrix}_q .$$

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Using Zeilberger's creative telescoping algorithm, it is possible to find a recurrence satisfied by our conjecture, and show that it is the same as on the previous slide.

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The only remaining thing to do is check that the initial conditions are also equal.

The generating functions as sums with obviously positive coefficients

$$G_{(5,0,0)}(z,q) = \sum_{n_1,n_2,n_3,n_4 \geq 0} \frac{z^{n_1} q^{n_1^2+n_2^2+n_3^2+n_4^2+n_1+n_2+n_3+n_4-n_1n_2+n_2n_4}}{(q;q)_{n_1}} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_4 \end{bmatrix}_q \begin{bmatrix} n_2 \\ n_3 \end{bmatrix}_q,$$

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$$G_{(4,0,1)}(z,q) = \sum_{n_1,n_2,n_3,n_4 \geq 0} \frac{z^{n_1} q^{n_1^2+n_2^2+n_3^2+n_4^2+n_1+n_3-n_1n_2+n_2n_4} \left(1+zq^{n_1+n_2+n_4+1}\right)}{(q;q)_{n_1}} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_4 \end{bmatrix}_q \begin{bmatrix} n_2 \\ n_3 \end{bmatrix}_q$$

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$$G_{(3,2,0)}(z,q) = \sum_{n_1,n_2,n_3,n_4 \geq 0} \frac{z^{n_1} q^{n_1^2+n_2^2+n_3^2+n_4^2+n_1-n_1n_2+n_2n_4} \left(q^{n_3}+zq^{n_1+1}+zq^{2n_1+n_3+2}+zq^{3n_1+n_2+n_3+n_4+3}\right)}{(q;q)_{n_1}} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q$$

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Our new A_2 Rogers–Ramanujan identities

Theorem (Corteel–D.–Uncu 2020)

We have

$$\sum_{n_1,n_2,n_3,n_4\geq 0} \frac{q^{n_1^2+n_2^2+n_3^2+n_4^2+n_1+n_2+n_3+n_4-n_1n_2+n_2n_4}}{(q;q)_{n_1}} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_4 \end{bmatrix}_q \begin{bmatrix} n_2 \\ n_3 \end{bmatrix}_q = \frac{1}{(q^2,q^3,q^3,q^4,q^4,q^5,q^5,q^6;q^8)_\infty},$$

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What next?

The following profiles are now well understood:

- all profiles of length 2 (Andrews–Gordon identities)
- all profiles of length 3 and sum 2 (Rogers–Ramanujan identities)
- all profiles of length 3 and sum 4 (Andrews–Schilling–Warnaar's A_2 Rogers–Ramanujan identities mod 7)
- all profiles of length 3 and sum 5 (our new A_2 Rogers–Ramanujan identities mod 8).

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We need to understand more profiles:

- Warnaar 2021: A_2 Andrews–Gordon identities, conjectures about the shape of the generating function for cylindric partitions of all profiles of length 3 and sum not divisible by 3
- profiles of length 3 and sum divisible by 3 seems to be the most difficult
- profiles with length > 3: still out of reach at the moment, but could lead to A_{n-1} Rogers–Ramanujan identities

Thank you very much!